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The dynamics of money velocity

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ABSTRACT
The dynamic of money velocity has changed gradually, influencing how macroeconomic shocks affect money velocity. This paper examines the response of money velocity to the external shocks in a system of equations, where money velocity, real output growth, money growth volatility, expected inflation, and risk premium are jointly determined. The regime-switching behaviour of money velocity is then examined through a Bayesian threshold approach. The main finding suggests that money velocity drops significantly after negative output and expected inflation shocks, followed by a sudden increase. In addition, the regime-switching process distinguishes the deterministic and stochastic behaviour of money velocity.

KEYWORDS
Hierarchical Bayesian estimation; inflation expectation; money growth; risk premium; velocity of money

JEL CLASSIFICATION
C11; E42; E51

I. Introduction

Money velocity, defined as the turnover rate of money supply, is the ratio of nominal GDP to money stock. It tends to move pro-cyclically, where its decline during the recession indicates the occurrence of fewer transactions. The behaviour of money velocity has been shifting over the last decades and has been intensified during the Covid-19 pandemic. A surge in money supply led to a rise in inflation and a significant decline in money velocity. These changes are studied theoretically and empirically. One view emphasizes the predictability and stability of money velocity (Chowdhury 1988; Friedman and Schwartz 1963; Hall and Noble 1987), whereas the other view underlines its stochastic nature (Pollin and Schaberg 1998; Gould and Nelson 1974; Serletis 1995).

This paper examines the response of money velocity to macroeconomic shocks within a Bayesian vector autoregressive model. The advantage of this Bayesian framework is that it addresses the over-parametrization problem that arises in traditional multivariate time series models. Over-parameterization, resulting from having a large number of parameters and a small number of observations, can be mitigated using informative priors in a hierarchical setting (Giannone, Lena, and Primiceri 2015; Koop and Korobilis 2010). First, using a Gibbs sampler, I estimate the posterior distributions by considering a simultaneous equation system that jointly determines money velocity, real output growth, money growth volatility, expected inflation, and risk premium.

To distinguish different macroeconomic regimes, I then examine the predictability and stability of money velocity through a Bayesian threshold autoregressive approach introduced and developed by Tong (1978, 1983), and Tsay (1989). This threshold model captures the dynamic behaviour of money velocity by switching the regimes given a threshold.

The findings suggest that money velocity responds to real output and expected inflation shocks in a U-shaped pattern, significantly reducing money velocity after negative shocks followed by a gradual increase. In addition, an initial rise and then decline in output growth is observed due to a positive expected inflation shock. This positive shock has a different impact on risk premium. The risk premium, which reveals the risk of unexpected change in the real interest rate, increases sharply after an expected inflation surprise but stabilizes over the intermediate term. The results also show the importance of distinguishing the regimes in

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studying the dynamics of money velocity. Money velocity can be predictable and stable under one regime and stochastic under another. This finding highlights the regime-switching behaviour of money velocity.

II. Money velocity responses to macroeconomic shocks

The monetary literature derives money demand function in general equilibrium frameworks with stochastic output and monetary expansion (Svensson 1985), explores stochastic properties of endogenous money velocity, inflation and interest rates based on cash-in-advance constraints (Hodrick, Kocherlakota, and Lucas 1991), and studies movements in money velocity in a dynamic general equilibrium setting (Palivos, Wang, and Zhang 1993). The appendix briefly reviews the theoretical background. The equation of exchange and velocity of money has motivated many empirical studies and estimation procedures (Evans 1984; Jung 2017; Hamilton 1989; Heller and Khan 1979; McGibany and Nourzad 1985; McMillin 1991).

This section addresses the over-parametrization problem when money velocity is jointly estimated with real output growth, money growth volatility, expected inflation and risk premium. Usually, joint estimation involves multivariate time series modelling, such as the vector autoregressive (VAR) models (Karfakis 1991; Raj 1995; Wang and Shi 2006). The VAR models include a large number of parameters which leads to over-parameterization. This problem is exacerbated when endogenous variables are more than two or three (Koop and Korobilis 2010). Bayesian methods have become popular to overcome over-parametrization. The VAR(p) model can be written as

$$y_t = \mathbf{C} + \sum_{i=1}^{p} \mathbf{A}_i y_{t-i} + \varepsilon_t, \tag{1}$$

where y_t is an $K \times 1$ vector containing observations on K endogenous variables, C is an $K \times 1$ vector of intercepts, A_i is an $K \times K$ matrix of coefficients, and ε_t is an $K \times 1$ vector of errors that is i.i.d $\mathcal{N}(0, \Sigma)$. We can obtain Bayesian estimators using a Gibbs sampler. Gibbs sampling belongs to the family of Markov Chain Monte Carlo (MCMC) methods for estimating posterior distributions from conditional distributions.

One way to improve the forecasting performance of VAR models is to combine the likelihood function with informative prior distribution (Doan, Litterman, and Sims 1984). In a hierarchical setting, we can use prior hyperparameters based on Bayes' law (Giannone, Lena, and Primiceri 2015). The prior distribution for the VAR coefficients can be written as

$$\mathbf{B}|\Sigma \sim \mathcal{N}(b, \Sigma \otimes \Omega\lambda), \tag{2}$$

where $B \equiv vec([C, A]^{'})$, and λ is a scalar parameter controlling the tightness of the prior. The conditional posterior can be obtained by multiplying the likelihood function by the prior. Hence, the posterior can be obtained by

$$p(\gamma|y) \propto p(y|\gamma)p(\gamma), \tag{3}$$

where $p(\gamma)$ denotes the prior density on the hyperparameters γ, known as *hyperprior*, while $p(y|\gamma)$ is the marginal likelihood, defined as the density of the data as a function of γ, and is written as

$$p(y|\gamma) = \int p(y|\theta, \gamma)p(\theta|\gamma)d\theta, \tag{4}$$

where θ is a vector of the autoregressive and variance parameters of the VAR model defined in equation (1). We can use the uninformative prior, but it leads to poor inference (Sims 1980). The common practice is to use conjugate priors in which the marginal likelihood equation (4) has a closed-form.

The commonly used priors belong to the normal-inverse Wishart family (Koop and Korobilis 2010). The conjugate prior belongs to the inverse Wishart distribution $\Sigma \sim \mathcal{IW}(\Psi, d)$. The Minnesota prior (Litterman 1980) imposes a random walk processes and is characterized by the following moments

$$\mathbb{E}[(\mathbf{A}_s)_{ij}|\Sigma] = \begin{cases} 1 & \text{if } i = j \text{ and } s = 1, \\ 0 & \text{otherwise}. \end{cases}$$

$$cov[(\mathbf{A}_s)_{ij}, (\mathbf{A}_r)_{kl}|\Sigma] = \begin{cases} \lambda^2 \frac{\Sigma_i k}{s^\alpha \psi_j / d-K-1} & \text{if } i=j \text{ and } s=1, \\ 0 & \text{otherwise}. \end{cases}$$

The posterior distribution approaches the prior when $\lambda \to 0$ and becomes closer to the sample information as $\lambda \to \infty$. See Chan et al. (2019), and Kuschnig and Vashold (2021) for details on applying Bayesian VAR with hierarchical priors.

The results of the Bayesian VAR model are presented next. Quarterly time series from January 1983 to July 2021 are used in the empirical study. The data are obtained from the Federal Reserve Bank of St. Louis and the U.S. Bureau of Economic Analysis. The velocity of money V_t is measured as the ratio of nominal GDP to M1 money stock. The growth rate of money velocity V^g is annualized and is calculated as the percent change from a year ago. The same method is used to calculate the growth rates for other variables. I include money growth volatility σ^m to address the point that decline in the velocity of money is due to an increase in the variance of money growth. In principle, higher money growth volatility raises the variability in the interest rates, resulting in an increased risk of holding bonds, higher demand for money, and a decline in money velocity (Evans 1984; Friedman 1984; Mascaro and Meltzer 1983). Following McMillin (1991), money growth volatility is constructed by an eight-quarter moving standard deviation. The growth rate of real GDP Y^g is also used as the real output measure. The literature also considers short- and long-term bond yields (Friedman and Schwartz 1982; Heller and Khan 1979; McMillin 1991). In addition, I include expected inflation and the risk premium. The expected rate of inflation π^e and real risk premium Δ^r are estimated over the next 30 years by the Federal Reserve Bank of Cleveland. The estimates result from models that incorporate Treasury yields, inflation data, inflation swaps, and survey-based measures of inflation expectations. Figure 1 plots the time series V^g, σ_m, Y^g, π^e, and Δ^r. The substantial drop in V^g and rise in money growth volatility due to the Covid-19 pandemic seem to question the classical dichotomy that emphasizes the independence of the monetary economy from the real economy. The threshold models explained in the next section aim to capture these regime shifts.

Table 1 presents the Bayesian VAR estimates and corresponding standard errors. The number of iterations for the Gibbs sampler is 10,000 with a burn-in of 5,000 and thinning of 15 observations. Figure 2 presents the trace and density plots of the marginal likelihood and hierarchically treated

Table 1. Hierarchical Bayesian VAR estimates.

	V^g	σ^m	Y^g	π^e	Δ^r
V^g_{t-1}	0.52	−0.08	−0.33	−0.59	−0.002
	(0.18)	(0.18)	(0.04)	(0.24)	(0.004)
σ^m_{t-1}	−0.41	1.52	−0.16	−0.42	−0.002
	(0.14)	(0.14)	(0.03)	(0.20)	(0.003)
Y^g_{t-1}	0.80	−0.12	1.35	0.42	0.013
	(0.45)	(0.44)	(0.12)	(0.58)	(0.013)
π^e_{t-1}	0.09	0.02	0.01	0.96	−0.001
	(0.05)	(0.05)	(0.01)	(0.08)	(0.001)
Δ^r_{t-1}	0.31	0.28	−0.28	0.11	0.439
	(0.95)	(0.96)	(0.72)	(1.01)	(0.145)
V^g_{t-2}	0.42	0.07	0.29	0.46	0.001
	(0.19)	(0.18)	(0.04)	(0.25)	(0.004)
σ^m_{t-2}	0.61	−0.54	0.20	0.49	0.001
	(0.17)	(0.16)	(0.03)	(0.23)	(0.003)
Y^g_{t-2}	−0.81	0.26	−0.46	−0.42	−0.012
	(0.44)	(0.44)	(0.11)	(0.57)	(0.013)
π^e_{t-2}	0.05	−0.07	0.01	−0.28	−0.001
	(0.06)	(0.05)	(0.01)	(0.08)	(0.001)
Δ^r_{t-2}	−0.11	0.06	0.46	0.24	−0.004
	(0.94)	(0.95)	(0.66)	(1.00)	(0.126)
c	−0.51	−0.12	0.19	−0.71	−0.004
	(0.61)	(0.62)	(0.17)	(0.82)	(0.019)

V^g, σ_m, Y^g, π^e, and Δ^r denote money velocity growth, money growth volatility, real GDP growth, expected inflation, and risk premium, respectively. Standard errors are reported in the parentheses.

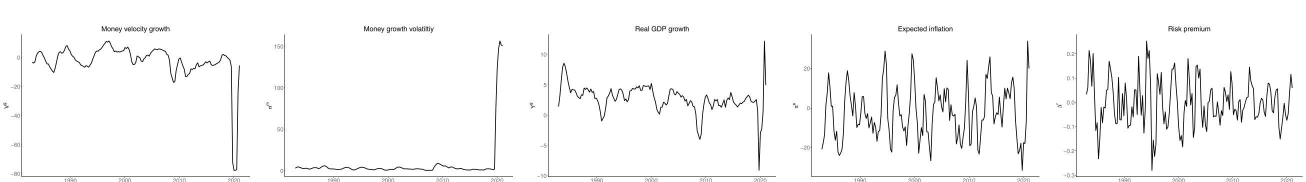

Figure 1. Time series plots of money velocity growth V^g (left), money growth volatility σ_m (middle left), real GDP growth Y^g (middle), expected inflation π^e (middle right), and risk premium Δ^r (right).

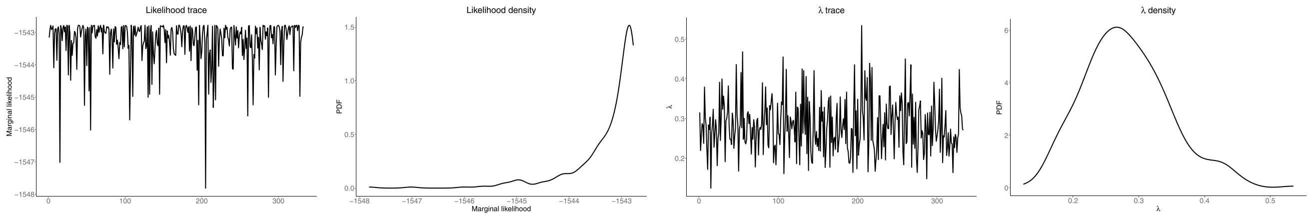

Figure 2. Trace and density plots of the marginal likelihood and hierarchically treated hyperparameter λ.

hyperparameter λ, showing the convergences. The mean of λ is .25. The trace plots also indicate a stationary pattern for the marginal likelihood and λ. The impulse response functions are given in Figure 3. The following points are noteworthy.

(1) The response of money velocity to real output growth and expected inflation shocks follow a U-shaped pattern. For the first few quarters, a significant reduction followed by an increase in money velocity is expected when output and expected inflation drop. The change in expected inflation is usually due to the change in money growth in response to prior economic shocks. Aligned with this finding, Benk, Gillman, and Kejak (2008) theoretically indicate a cyclical increase in the velocity of money due to monetary shocks. Alvarez, Atkeson, and Edmond (2009) also show that price level responds to a rise in money supply, offsetting endogenous reduction in velocity.

(2) An expected inflation surprise results in a temporary rise in output, followed by an intermediate-term decline. The economy, however, would recover afterwards. This finding is consistent with the expected inflation channel, which states that output rises because of an increased consumption (Christiano, Eichenbaum, and Rebelo 2011). Expected inflation rises sharply for two quarters after an output surprise, but it drops gradually.

(3) The yield spread widens sharply due to an expected inflation surprise but stabilizes after a few quarters. The literature on the relationship between the risk premium and macroeconomic factors is vast. A few studies suggest the real short rate is negatively correlated with expected inflation (Ewing 2003; Ang, Bekaert, and Wei 2008). A negative output shock leads to a sudden reduction in risk premium. However, this effect disappears over time.

The dynamic behaviour of money velocity

The behaviour of the velocity of money changes over time. Therefore, models with fixed coefficients would not capture the dynamics since changes require varying coefficients. Instead, a threshold model captures the dynamics of money velocity within different regimes when the values of a variable exceed a certain threshold. Threshold models are a special case of regime-switching models. Threshold autoregressive models (TAR) are proposed and discussed by Tong (1978, 1983). These models capture the changes by switching the regimes.

Let y_{t-d} denote the threshold variable, where d is the delay parameter. The TAR model denoted by TAR(j, p) can be written as

$$y_t = \phi_0^{(j)} + \sum_{i=1}^{p} \phi_i^{(j)} y_{t-i} + \theta^{(j)} \varepsilon_t \quad \text{for} \quad y_{t-d} \in R^j, \tag{5}$$

where j are indicator random variables, taking integer values in $\{1, \cdots, J\}$ and can be estimated using a Markov chain driven TAR model (Tong and Lim 1980; Tsay 1989). The threshold variable $y_{t-d} \in R^j$ is the j^{th} regime of the TAR model. Markov switching models are obtained when j is hidden (Hamilton 1989).

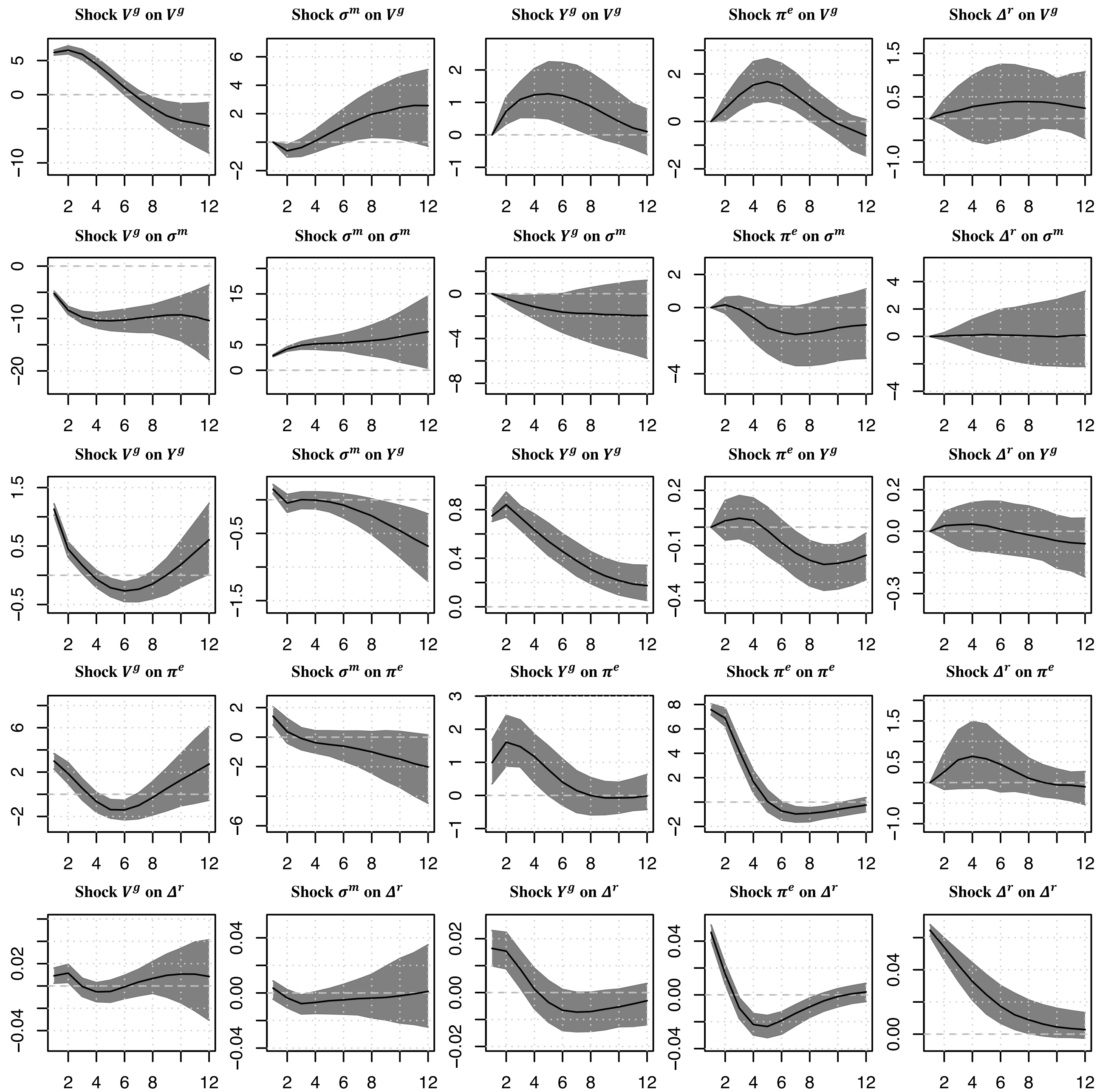

Figure 3. Response of V^g, σ_m, y^g, π^e, and Δ^r to V^g shock (left), σ_m shock (middle left), Y^g shock (middle), π^e shock (middle right), and Δ^r shock (right). V^g, σ_m, y^g, π^e, and Δ^r denote money velocity growth, money growth volatility, real GDP growth, expected inflation, and risk premium. Confidence bands are given at 95%.

The statistical inference procedure of the TAR models is shown by Hansen (1997). We can estimate the parameters of the two-regime TAR models by ordinary least squares, where the least-squares estimators are consistent and converge almost surely (Tsay 1989). A TAR (2, 2) process can be written as

$$y_t = \begin{cases} \phi_0^{(1)} + \sum_{i=1}^{p_1} \phi_i^{(1)} y_{t-i} + \varepsilon_t^{(1)} & y_{t-d} \le r, \\ \phi_0^{(2)} + \sum_{i=1}^{p_2} \phi_i^{(2)} y_{t-i} + \varepsilon_t^{(2)} & y_{t-d} > r, \end{cases} \tag{6}$$

where r is the threshold parameter. Table 2 presents the results of the Bayesian TAR model. The Bayesian TAR coefficients for regime j are

Table 2. Bayesian threshold autoregressive estimates.

	Mean	Median	SD	Lower	Upper
$\phi_0^{(1)}$	−1.93	−1.91	1.23	−4.34	0.54
$\phi_1^{(1)}$	1.10	1.10	0.11	0.88	1.33
$\phi_2^{(1)}$	−0.36	−0.36	0.11	−0.59	−0.14
$\phi_0^{(2)}$	−0.17	−0.17	0.31	−0.78	0.44
$\phi_1^{(2)}$	0.98	0.98	0.06	0.85	1.11
σ_1^2	89.83	88.39	15.24	64.99	125.08
σ_2^2	2.87	2.82	0.46	2.11	3.89
r	−1.44	−1.51	0.19	−1.66	−1.00
DIC = 554.01 r acceptance rate = 76.42%					

Bayesian TAR estimates for regime j are represented by $\phi_i^{(j)}$. The threshold coefficient is denoted by r. Each regime is an i.i.d Gaussian white noise process with mean 0 and variance σ_j^2.

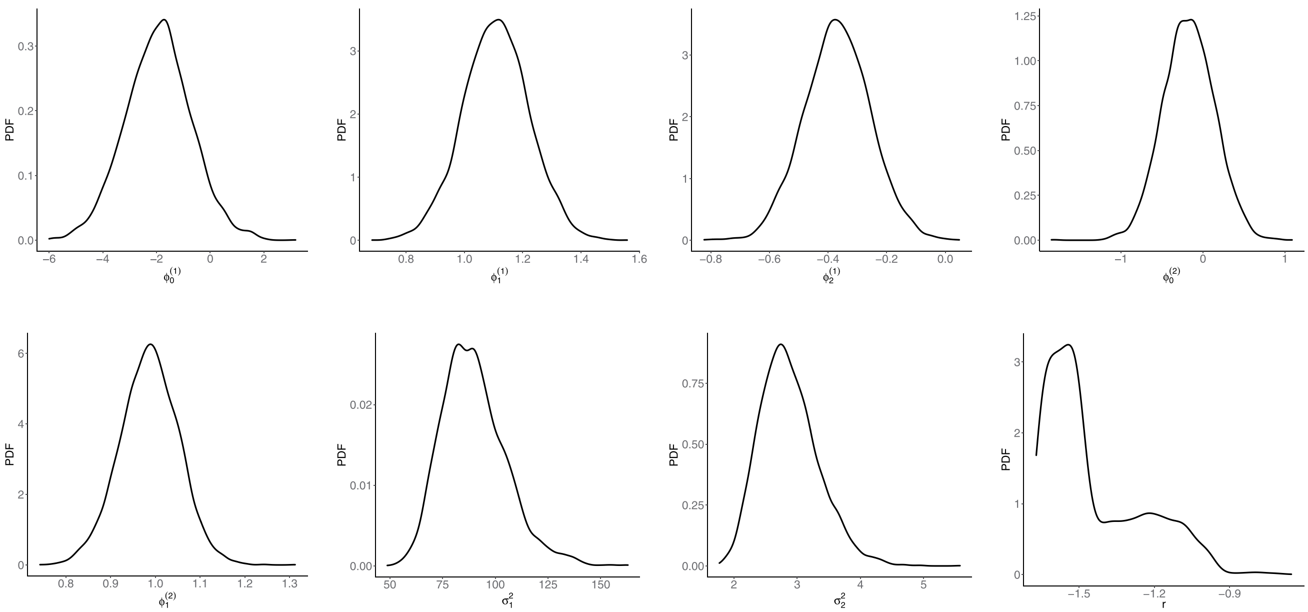

Figure 4. Posterior distributions for the Bayesian TAR estimates. Estimates for regime j are represented by $\phi_i^{(j)}$. The threshold parameter is denoted by r. Each regime is an i.i.d Gaussian white noise process with mean 0 and variance σ_j^2.

represented by $\phi_i^{(j)}$. The threshold coefficient is denoted by r. Each regime follows an i.i.d Gaussian white noise process with mean 0 and variance σ_j^2. MCMC trace plots and autocorrelation functions of the estimates indicate that the MCMC samples are well mixed and converged. Figure 4 presents the posterior distributions. The threshold coefficient is −1.44. The finding suggests that money velocity is predictable under Regime 1 and follows a random walk under Regime 2. This finding highlights the regime-switching behaviour of money velocity.

III. Concluding remarks

The velocity of money can be influenced by factors such as individuals' habits of saving and consumption and payment systems Fisher (1911b). However, the equation of exchange has been considered as the basis of determining the behaviour of money velocity, indicating the stability of velocity due to the stability of money demand. As opposed to the view that money velocity grows at a constant rate, there is no reason to assume constant velocity. Hence, the velocity of money changes depending on social habits, the banking system, and the degree of financial sophistication. This paper examines how the behaviour of money velocity changes over time by first addressing endogeneity in the velocity equation and then studying the regime-switching behaviour. The findings illustrate money velocity responses to macroeconomic shocks and indicate its dynamics. Money velocity moves pro-cyclically. During recession interest rates, the opportunity cost of holding money, fall and velocity declines.

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Appendix: Theoretical Background

In the 1950s, 60s, and 70s, money velocity experienced stable growth that led to the adoption of monetary aggregate targets by the Federal Reserve (Chowdhury 1988). The predictability of the velocity of money during this period has been studied extensively. The seminal work of Friedman and Schwartz (1963) supported by others, such as Hall and Noble (1987) highlights the stability of money velocity. However, the predictable link between money growth and nominal GDP has disappeared over time, and the stability of money velocity has been questioned after its decline and high volatility since 1980. A few empirical studies showed the stochastic behaviour of money velocity and that the velocity of money follows a random walk process (Gould and Nelson 1974; Serletis 1995). Theoretical work on monetary velocity started with the study of demand for money, general equilibrium approaches, and cash-in-advance models, where the money velocity is variable (Hodrick, Kocherlakota, and Lucas 1991; Palivos, Wang, and Zhang 1993; Svensson 1983, 1985).

The determinants of money velocity can be examined in a dynamic general equilibrium model with a cash-in-advance constraint. An agent's utility U is defined as

$$U = \sum_{t}^{\infty} \beta^t u(c_t), \tag{7}$$

where $\beta \in (0, 1)$ is the discount rate, c_t is per capita consumption, and U is increasing and strictly concave. The agent produces $A_t f(k_t)$, where k_t is capital and A_t is the technological parameter, and seeks to maximize the utility function subject to the budget constraint

$$c_t + m_{t+1} + k_{t+1} = A_t f(k_t) + \frac{m_t}{1+\pi_t} + \tau_t, \tag{8}$$

where m_t, π_t, and τ denote real money holdings, inflation, and the lump sum cash transferred from the government. The cash-in-advance (liquidity) constraint states that the agent purchases all consumption and a fraction θ of investment goods in cash. Hence the cash-in-advance constraint can be written as

$$c_t + \theta(\pi, \phi_t) k_{t+1} \leq \frac{m_t}{1+\pi_t} + \tau_t, \tag{9}$$

where ϕ is an exogenous credit enhancement measure. The government, on the other hand, has the budget constraint

$$\tau_t = m_{t+1} - \frac{m_t}{1+\pi_t}. \tag{10}$$

The objective function (7) is maximized subject to budget and cash-in-advance constraints defined in (8) and (9), combining with the government budget constraint (10), resulting in the goods market equilibrium

$$c_t + k_{t+1} = A_t f(k_t). \tag{11}$$

The velocity of money, is hence theoretically defined as

$$V_t \equiv \frac{c_t + k_t}{m_t} = \frac{c_t + k_t}{c_t + \theta k_t}. \tag{12}$$

See Palivos, Wang, and Zhang (1993) for details. The velocity equation (12) indicates higher θ and m_t leads to lower money velocity. Also, money growth negatively affects velocity by impacting θ and m_t. Higher money growth is associated with higher inflation. Higher inflation leads to a decline in investment, resulting in a decline in money velocity. From (12), real output can be defined as the sum of consumption and capital $y_t = c_t + k_t$ and hence, money velocity is a function of θ, money growth, inflation, and output.

Money velocity can be also derived from the money demand function and the equation of exchange introduced by Fisher (1911a, 1911b). Real demand for money $m^d = f(Y_t, r_t) = \alpha Y_t^{\gamma_1} r_t^{\gamma_2}$, where r_t is the real interest rate. In equilibrium, $m^d = m^s$, where m^s is real money supply. From the equation of exchange, the money velocity can be written as

$$V_t = \frac{Y_t}{\alpha Y_t^{\gamma_1} r_t^{\gamma_2}} = \alpha^{-1} Y_t^{1-\gamma_1} r_t^{-\gamma_2}. \tag{13}$$

The log-linear form can be written as

$$\log V_t = -\log \alpha + (1-\gamma_1)\log Y_t - \gamma_2 \log r_t. \tag{14}$$

Hence, we can consider the money velocity equation as

$$V_t = f(Y_t, m_t^s, \pi_t, r_t). \tag{15}$$